JMAT 7303

Degree Master of Science in Mathematical Modelling and Scientific Computing

Mathematical Methods II

TRINITY TERM 2014

Thursday, 24th April 2014, 9:30 a.m. – 11:30 a.m.

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.

All questions will carry equal marks.

Do not turn over until told that you may do so.

Section A — Applied Partial Differential Equations

Question 1

(a) Suppose that the functions $p(r)$ and $q(r)$ satisfy

$$F(p(r), q(r)) \equiv 0,$$

for all r, where the continuously differentiable function $F : \mathbb{R}^2 \to \mathbb{R}$ is known. Suppose also that the functions $x(r, s)$, $y(r, s)$ and $u(r, s)$ are given by

$$x(r, s) = x_0(r) + s\frac{\partial F}{\partial p}, \quad y(r, s) = y_0(r) + s\frac{\partial F}{\partial q}, \quad u(r, s) = u_0(r) + s\left(p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}\right),$$

where $x_0(r)$, $y_0(r)$ and $u_0(r)$ satisfy the equation

$$\frac{du_0}{dr} \equiv p(r)\frac{dx_0}{dr} + q(r)\frac{dy_0}{dr},$$

for all r. Show that

$$\frac{\partial u}{\partial r} = p\frac{\partial x}{\partial r} + q\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial u}{\partial s} = p\frac{\partial x}{\partial s} + q\frac{\partial y}{\partial s}.$$

Deduce that $p = \partial u/\partial x$ and $q = \partial u/\partial y$ provided

$$\frac{\partial x}{\partial s}\frac{\partial y}{\partial r} - \frac{\partial x}{\partial r}\frac{\partial y}{\partial s} \neq 0.$$

[10 marks]

(b) Find the solution $u(x, y)$ of the partial differential equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1,$$

with $u(x, 1) = x^2$ for $x > 0$. Sketch the region in which your solution is uniquely defined.

[15 marks]

Question 2

(a) Let $A(x, y)$ and $B(x, y)$ be $(n \times n)$ matrices and let $u(x, y) \in \mathbb{R}^n$ be continuously differentiable. Suppose that for $(x, y) \in \mathbb{R}^2$

$$A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} = c \quad \text{for} \;\; c(x, y) \in \mathbb{R}^n.$$

State conditions on A and B that make this system of partial differential equations hyperbolic and define the characteristic variables for the system. **[7 marks]**

(b) Let

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

Find characteristic variables and Riemann invariants for this system. Use your results to find the solution $u(x, y) = (u, v)^T$ when the initial data is

$$u(0, y) = 6\cos y + e^{-y}, \quad v(0, y) = \cos y - e^{-y}.$$

[14 marks]

(c) Explain where your solution is uniquely determined by the initial data and include a sketch of this region of the (x, y)-plane.

[4 marks]

Question 3

(a) Suppose that $n(x,t) > 0$ solves

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x}\left(n\frac{\partial n}{\partial x}\right) + \frac{n}{2t}, \tag{1}$$

in a region bounded by the lines $t = 0$, $t = \tau$, and two, non-intersecting, smooth curves C_1 and C_2. Prove that if a solution exists then $n(x,t)$ attains its minimum value on $t = 0$ or on one of the curves C_1 or C_2.

[6 marks]

(b) Let Ω be the region bounded by the curves $x = 0, x = L(t)$ and $t = 0$ so that

$$\Omega = \{(x,t) : 0 < x < L(t) \text{ and } 0 < t\}.$$

Suppose further that $n(x,t)$ and $L(t)$ solve equation (1) in Ω, with

$$n(0,t) = 1, \quad \frac{\partial n}{\partial x}(0,t) = 0, \quad n(L(t),t) = 0.$$

Determine the real constants α and β and the function $N \in C^2$ for which there exists a solution of the form $n(x,t) = t^{\alpha}N(x/t^{\beta})$. Derive analytical expressions for $n(x,t)$ and $L(t)$.

[10 marks]

(c) Suppose now that $n(x,t)$ and $a(x,t)$ solve the following, coupled problem in the domain $\hat{\Omega} = \{(x,t) : 0 < x < \hat{L}(t),\ 0 < t\}$:

$$\left.\begin{aligned} \frac{\partial n}{\partial t} &= \frac{\partial}{\partial x}\left(n\frac{\partial n}{\partial x} - \chi n\frac{\partial a}{\partial x}\right) + \frac{n}{2t} \\ 0 &= \frac{\partial^2 a}{\partial x^2} - \frac{1}{t} \end{aligned}\right\}$$

where $0 < \chi < 1/2$ is a constant, and

$$n(0,t) = 1, \quad \frac{\partial n}{\partial x}(0,t) = 0, \quad n(\hat{L}(t),t) = 0,$$

$$\frac{\partial a}{\partial x}(0,t) = 0, \quad a(\hat{L}(t),t) = 1.$$

Determine the functions $N, A \in C^2$ for which there exist solutions of the form

$$n(x,t) = t^{\alpha}N\left(\frac{x}{t^{\beta}}\right), \quad a(x,t) = t^{\alpha}A\left(\frac{x}{t^{\beta}}\right).$$

Use your results to explain briefly how the speed with which the leading front $x = \hat{L}(t)$ moves changes as $\chi \in (0,1/2)$ increases.

[9 marks]

Question 4

(a) Suppose that $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial Q}{\partial x} = 0, \tag{2}$$

in $-\infty < x < \infty$ and $0 < t$, where $Q = Q(u)$ is a positive function of u, with $Q'(u) > 0$ and $Q''(u) > 0$ for all values of u. Suppose further that $u(x,0) = u_0(x)$.

Construct an implicit solution for $u(x,t)$ and explain briefly why the solution represents a wave moving to the right. Show further that if $u_0'(s) < 0$ for some value of s then a shock will form when $t = t^* > 0$ where

$$t^* = \min_s \left\{ \frac{1}{|u_0'(s)| Q''(u_0(s))} \right\}.$$

[9 marks]

(b) Suppose that $Q(u) = u^2/2$ and

$$u_0(x) = \begin{cases} 0 & x \leqslant -1, \\ 1 - |x| & -1 < x \leqslant 1, \\ 0 & 1 < x. \end{cases}$$

Find $u(x,t)$ explicitly and show that for $0 < t \ll 1$, $u(x,t) = u_0(x) + O(t)$. Determine the time $t = t^* > 0$ at which a shock forms. Sketch the behaviour of $u(x,t)$ as a function of x as t increases from $t = 0$ to $t = t^*$. **[8 marks]**

(c) For $t > t^*$, let the shock position be given by $x = X(t)$. Use the Rankine-Hugoniot condition to determine $X(t)$. Indicate the path of the shock on a sketch of the characteristic curves. **[8 marks]**

Section B — Further Applied Partial Differential Equations

Question 5

The Bessel functions $J_m(x)$ are the solutions of the equation

$$\left[x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+x\frac{\mathrm{d}}{\mathrm{d}x}+x^2-m^2\right]J_m(x)=0.$$

that are bounded as $x\to 0$.

(a) Use the generating function

$$\phi=\sum_{m=-\infty}^{\infty}t^m J_m(x)=\exp\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\}$$

to establish the results

$$2\frac{\mathrm{d}}{\mathrm{d}x}J_m(x)=J_{m-1}(x)-J_{m+1}(x),\qquad \frac{2m}{x}J_m(x)=J_{m-1}(x)+J_{m+1}(x).$$

Hence show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x^m J_m(x)\big)=x^m J_{m-1}(x).$$

[6 marks]

(b) A continuous function $f(x)$ in the interval $0\le x<X$ may be expanded as the series

$$f(x)=\sum_{n=1}^{\infty}c_n J_1(k_n x),$$

where the constants $0<k_1<k_2<k_3<\cdots$ are given by the roots $J_1(k_n X)=0$.
Show that the coefficients c_n are given by

$$c_n=\frac{2}{[XJ_1'(k_nX)]^2}\int_0^X f(x)J_1(k_nx)\,x\,dx.$$

[6 marks]

You may use the identity $J_m(0)=0$ for $m\ge 1$, and the relations

$$\int_\alpha^\beta x\,J_m(kx)J_m(\ell x)\,\mathrm{d}x=\frac{1}{k^2-\ell^2}\Big[\ell xJ_m(kx)J_m'(\ell x)-kxJ_m(\ell x)J_m'(kx)\Big]_\alpha^\beta \text{ for } k,\ell>0 \text{ with } k\ne\ell,$$

and

$$\int_\alpha^\beta x\,(J_m(kx))^2\,\mathrm{d}x=\frac{1}{2}\left[\left(x^2-\frac{m^2}{k^2}\right)(J_m(kx))^2+x^2(J_m'(kx))^2\right]_\alpha^\beta \text{ for } k>0.$$

(c) Hence show that

$$x = -2\sum_{n=1}^{\infty} \frac{J_1(k_n x)}{k_n J_0(k_n)} \quad \text{for} \quad 0 \le x < 1.$$

[11 marks]

(d) What is the value of this series at $x = 1$?

[2 marks]

Question 6

(a) Write down the forward Hankel transform of a function $u(r)$ with respect to r, and the corresponding inverse transformation.

[3 marks]

(b) Derive the expression for the forward Hankel transform from the two-dimensional Fourier transform.

[7 marks]

(c) Small horizontal displacements $u(z,t)$ of a heavy vertical chain are governed by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial z^2} + \frac{1}{z}\frac{\partial u}{\partial z}.$$

The chain is initially at rest, hanging vertically downwards. It is hit with a hammer so that $u_t(z,0) = \epsilon$ for $0 \leq z \leq a$, while $u_t(z,0) = 0$ for $z > a$. Show that the solution for $t > 0$ may be written as

$$u(z,t) = \epsilon \int_0^\infty \frac{a}{k} J_1(ka) J_0(kz) \sin(kt) \, \mathrm{d}k.$$

[15 marks]

You may use the integral representation $J_m(x) = (2\pi)^{-1} \int_0^{2\pi} \exp\{\mathrm{i}(m\phi - x\cos\phi)\} \, \mathrm{d}\phi$ *and the recurrence relation* $xJ_m'(x) + mJ_m(x) = xJ_{m-1}(x)$ *without proof.*